

Geometry I, 2006 : Coursework 10 Solutions

1. [2 marks] Let t be a linear transformation of $\mathbb{R}^n$. We call $v \in \mathbb{R}^n$ an *eigenvector* of t if $v \neq 0_n$ and $t(v) = \lambda v$ for some scalar λ , in which case λ is called the *eigenvalue* of t corresponding to v .

2. [3 marks]

$$\begin{aligned} t(w) &= t(\alpha u + \beta v) = t(\alpha u) + t(\beta v) = \alpha t(u) + \beta t(v) \\ &= \alpha(\lambda u) + \beta(\lambda v) = \lambda(\alpha u) + \lambda(\beta v) = \lambda(\alpha u + \beta v) = \lambda w. \end{aligned}$$

3. (i) Let $A = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$.

[1 mark] Then $Av = -2v$ for all $v \in \mathbb{R}^2$, and so the only eigenvalue of A is -2 .

[1 mark] Moreover, the set of all eigenvectors of A consists of all the elements of $\mathbb{R}^2$ except for the zero vector, that is, $\mathbb{R}^2 \setminus \{0_2\}$, equivalently

$$\left\{ v \in \mathbb{R}^2 : v \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

(ii) Let $A = \begin{pmatrix} 2 & 9 \\ 4 & -3 \end{pmatrix}$.

$$\text{Then } \det(A - xI_2) = \begin{vmatrix} 2-x & 9 \\ 4 & -3-x \end{vmatrix} = (2-x)(-3-x) - 36 = x^2 + x - 42 = (x+7)(x-6).$$

[1 mark] Thus the eigenvalues are -7 and 6 .

To determine the eigenvectors with corresponding eigenvalue -7 , we solve

$$(A + 7I_2) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{ i.e. } \begin{pmatrix} 9 & 9 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is equivalent to the single equation $x + y = 0$. Thus y can be any real number, r say, and then $x = -r$.

[1 mark] Thus, the set of all eigenvectors with corresponding eigenvalue -7 is

$$\left\{ \begin{pmatrix} -r \\ r \end{pmatrix} : 0 \neq r \in \mathbb{R} \right\}.$$

To determine the eigenvectors with corresponding eigenvalue 6 , we solve

$$(A - 6I_2) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{ i.e. } \begin{pmatrix} -4 & 9 \\ 4 & -9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is equivalent to the single equation $-4x + 9y = 0$. Thus y can be any real number, r say, and then $x = \frac{9}{4}r$.

[1 mark] Thus, the set of all eigenvectors with corresponding eigenvalue 6 is

$$\left\{ \begin{pmatrix} \frac{9}{4}r \\ r \end{pmatrix} : 0 \neq r \in \mathbb{R} \right\}.$$

(iii) Let $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$.

Then $\det(A - xI_2) = \begin{vmatrix} 3-x & 1 \\ 0 & 3-x \end{vmatrix} = (3-x)(3-x) = (x-3)^2$.

[1 mark] Thus the only eigenvalue is 3.

To determine the eigenvectors, we solve

$(A - 3I_2) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, i.e. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This is equivalent to the single equation $y = 0$. Thus x can be any real number, r say, and $y = 0$.

[1 mark] Thus, the set of all eigenvectors with corresponding eigenvalue 3 is

$$\left\{ \begin{pmatrix} r \\ 0 \end{pmatrix} : 0 \neq r \in \mathbb{R} \right\}.$$

4. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 4 \end{pmatrix}$.

The characteristic polynomial of A is

$$\begin{aligned} \det(A - xI_3) &= \begin{vmatrix} 1-x & 2 & 0 \\ 2 & 1-x & 0 \\ 1 & 2 & 4-x \end{vmatrix} = \begin{vmatrix} 0 & 1-x & 2 \\ 0 & 2 & 1-x \\ 4-x & 1 & 2 \end{vmatrix} \\ &= 0 - 0 + (4-x) \begin{vmatrix} 1-x & 2 \\ 2 & 1-x \end{vmatrix} \\ &= (4-x)((1-x)^2 - 4) = -(x-4)(x^2 - 2x - 3) = -(x-4)(x-3)(x+1). \end{aligned}$$

[1 mark for the correct characteristic polynomial]

[1 mark] Hence, the eigenvalues are $-1, 3, 4$.

To determine an eigenvector with corresponding eigenvalue -1 , we solve

$(A + I_3) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, i.e. $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, which is equivalent to the following system of linear equations:

$$\begin{cases} 2x + 2y & = 0 \\ 2x + 2y & = 0 \\ x + 2y + 5z & = 0 \end{cases}.$$

This system is equivalent to

$$\begin{cases} x + 2y + 5z & = 0 \\ 2x + 2y & = 0 \end{cases},$$

which is equivalent to

$$\begin{cases} x + 2y + 5z = 0 \\ -2y - 10z = 0 \end{cases}.$$

Taking $z = 1$, we have $y = -5z = -5$, and $x = -2y - 5z = 10 - 5 = 5$.

[1 mark] Thus, $\begin{pmatrix} 5 \\ -5 \\ 1 \end{pmatrix}$ is an eigenvector with corresponding eigenvalue -1 .

Similar calculations determine that

[1 mark] $\begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$ is a eigenvector with corresponding eigenvalue 3 , and

[1 mark] $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector with corresponding eigenvalue 4 .

5. [3 marks] Suppose A is an $n \times n$ matrix such that $A^2 = 0_{nn}$ and λ is an eigenvalue of A . Then there is a $v \in \mathbb{R}^n$ such that $v \neq 0_n$ and $Av = \lambda v$. We thus have

$$0_n = 0_{nn}v = A^2v = A(Av) = A(\lambda v) = \lambda(Av) = \lambda(\lambda v) = \lambda^2v.$$

Since $\lambda^2v = 0_n$, but $v \neq 0_n$, we must have $\lambda^2 = 0$, and so $\lambda = 0$.

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